GUIDE #7

LEARNING IN UNDERGRADUATE MATHEMATICS: THE OUTCOME SPECTRUM (LUMOS) "HOW TO" GUIDES

Develop Mathematical Habits

BILL BARTON

TEACHING & LEARNING RESEARCH INITIATIVE
NĀU I WHATU TE KĀKAHU, HE TĀNIKO TAKU
AKO AOTEAROA
NATIONAL CENTRE FOR TERTIARY TEACHING EXCELLENCE
NPF NATIONAL PROJECT FUND
AKO AOTEAROA
THE UNIVERSITY OF AUCKLAND
Te Whare Wānanga o Tāmaki Makaurau
NEW ZEALAND

A series of "How to" guides

"HOW TO" GUIDE #7: This guide is one of seven produced by the project Learning in Undergraduate Mathematics: The Outcome Spectrum (LUMOS). LUMOS examined the learning outcomes of undergraduates in the mathematical sciences.

The full list of titles in the series is:

- "How to" Guide #1: Implement team-based learning
- "How to" Guide #2: Implement semi-authentic mathematical experiences
- "How to" Guide #3: Shift responsibility for learning onto students
- "How to" Guide #4: Monitor feelings and beliefs about the mathematical sciences
- "How to" Guide #5: Monitor the development of mathematical communication
- "How to" Guide #6: Generate conceptual readiness
- "How to" Guide #7: Develop mathematical habits

LEARNING IN UNDERGRADUATE MATHEMATICS: THE OUTCOME SPECTRUM (LUMOS).

"HOW TO" GUIDE #7: DEVELOP MATHEMATICAL HABITS

Authors
Bill Barton

This work was supported through the Teaching and Learning Research Initiative (TLRI) and Ako Aotearoa National Project Fund 2012. A series of "How to" guides and the research report can be downloaded at www.tlri.org.nz/undergraduatemathematics or akoaotearoa.ac.nz/undergraduate-mathematics. The guides are also available for purchase at www.shop.akoaotearoa.ac.nz.

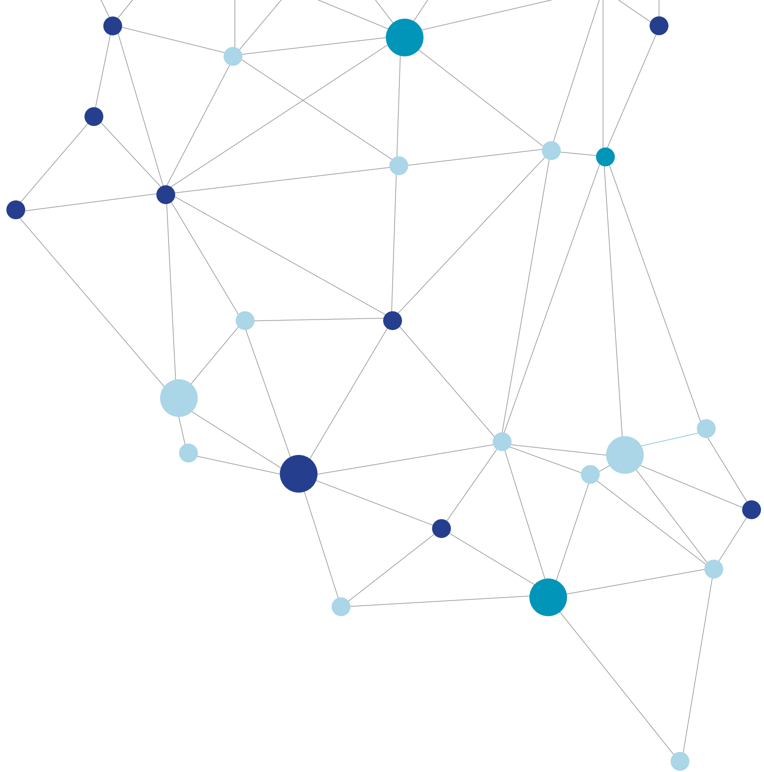
Publisher
Ako Aotearoa – The National Centre for Tertiary Teaching Excellence
PO Box 756
Wellington 6140

Published
August 2017

ISBN 978-0-947516-59-8 (online)
ISBN 978-0-947516-60-4 (print)

Design by Two Sparrows, www.twosparrows.co.nz

Contents



Mathematical habits

The importance of developing good mathematical habits at undergraduate level was emphasised both through our interviews with mathematicians, and the research literature. Furthermore, lists of habits are available. They are also called mathematical modes, processes, behaviours, or attributes.

The habits come in different forms, from the very mathematics specific (like proving) to the more general habits that could apply to many disciplines (like persisting). (See page 8 for a catalogue of habits). But nearly every mathematician we spoke to mentioned more than one habit specifically. Several said that the development of these habits was at least as important, or more important, than learning any specific content as an outcome of undergraduate learning.

Despite this affirmation of importance, very few mathematicians reported explicitly attempting to teach any of these habits, except for mathematical proving and mathematical modelling. None reported trying to observe these habits in development.

In our project, we also had little success, despite several attempts to observe these behaviours. So, this guide is a call to the undergraduate teaching community to try to work together on what we agree is a vital part of our students' learning.

Through our attempts to observe habits, we have come to some conclusions about why it is so difficult, and a few proposals as to how these difficulties may be overcome. We even made some progress.

Section 2 (Difficulties of observation) specifies the three main underlying difficulties, illustrating them from our experiences within the project and the literature. Section 3 (Observing mathematical habits) describes the progress we made. This is not intended to be a programme to follow, but more an example of what needs to be done. This section also highlights our catalogue of habits.

Difficulties of observation

We encountered three sources of difficulty when attempting to observe the development of mathematical habits at undergraduate level. They were:

- Defining or describing the habit accurately enough.
- The relative rarity of instances of each habit, especially at an individual level.
- The dependence of performance on mathematical content knowledge.

Describing the habit

Mathematical habits take many forms depending on the mathematical content and level, the stage of development of the habit, and the wider context in which it occurs. For example, proof is different in geometry or in number theory, at graduate level or at the edge of research, for primary students or for undergraduates, in a tutorial or in a research presentation. Proofs may be verbal explanations, diagrams, symbolic statements, computer programmes, or logical prose—and is usually some combination of these.

Of course, there is a common essence, but describing that in a way that meets general agreement was more difficult than expected. However, our description needed to be more than that. We had to have a description that was sufficiently detailed, so that it contained at least one item that was a behaviour that could be observed.

One approach is to give the different instances of proof different names: reasoning, justification, argument, proof. This is particularly common when talking about the development of the habit: justification that does not meet some lecturer's formal expectations may not be given the name "proof".

Our view is that a habit needs to be described both as it appears in various stages of development, and also in its fully-fledged state, in all its forms (i.e. inclusively). For example, part of learning what a proof is, is learning when different forms of proof are appropriate or when the form itself needs justification.

With respect to the stages of development, our experience is that this is not normally a linear set of stages, a ladder to be climbed. Students seem to learn habits in many different ways, through different sets of experiences. It is more like a climbing frame: there are many different routes to the top, with the climber utilising different skills or experiences depending on the route.

Another example of the difficulty in describing a habit occurred when we attempted to describe "deep mathematics". This phrase was used often by mathematicians when explaining what they hoped students would bring to their courses: the lecturer was not so concerned about pre-requisite content so much as a positive approach and some experience of "a deep experience of mathematics". What was this experience?

At first, we discussed some of the usual mathematical processes (proving, hypothesising, generalising, abstracting, symbolising) and wondered whether a deep experience was a combination of these. Then we focused on abstraction. We agreed that abstraction was a key process, a sine qua non, of deep mathematics. We agreed that a hallmark of abstraction was "reasoning with concepts". At this point we had reached a descriptor that identified a specific behaviour that we could observe.

Paucity of instances

We consider it an open question as to whether, and how, course delivery might change to elicit more instances of mathematical processes in students. However, in standard courses, there seem to be few opportunities for students to engage in mathematical processes at all, let alone on occasions where their behaviour can be observed.

Consider the process of hypothesising, for example. In most cases where students might be asked to predict a mathematical relationship or anticipate the outcome of a mathematical procedure, the "true" answer is already known. Authentic hypothesising in a situation where the outcome is unknown, or when a mathematical situation is being explored, is not a standard experience for undergraduates.

The consequence of having few instances is, of course, an inability to identify stages of development in the process. In cases like hypothesising, the instances are so few that even if we could observe and record a whole class during all tutorials in a course, we are still unlikely to have a sufficient collection of instances to judge any developments.

One of the other habits we identified was that of mathematical foresight: knowing what is likely to happen if a certain pathway is taken, and/or having a general overview of what a solution will look like and a broad trajectory of a likely path to get there. This is another habit where very few instances occur during an undergraduate course.

In cases like these, our only option seems to be to construct special opportunities in, for example, a tutorial. However, we are unconvinced that such an approach is either useful for the students, or would give us a valid picture of how well a class is progressing with the mathematical habit.

Dependence on content

It is no surprise that mathematical habits should be dependent on content knowledge. A student will be better at proving if they know more about the topic, their hypotheses will be more often accurate or more detailed, their ability to persist will be more robust because they know more possible pathways, and their use of visual representations will be more varied and more useful.

How, then, can we separate the habit from the mathematical knowledge? We suspect that many undergraduate lecturers assume that the development of mathematical habits happens spontaneously as students know more and get more experienced. This may be why very few attempts have been made to explicitly teach or to observe the learning of habits.

However, some of our attempts at observation have overcome the dependence on content.

One technique to overcome dependence on content is to provide all the needed content at the time that the habit is being observed. For example, at the end of a topic, a question could be posed that asks students to hypothesise the conditions under which a certain relationship would hold. All the information required to make a sensible hypothesis would have been given in the previous lectures or available in notes or texts, but the particular relationship would not have been previously discussed. Such a question is not too unlike many examination questions, except that the required response is a hypothesis that is judged not on its truth or correctness, but on its appropriateness as a hypothesis.

A second technique is to present a situation that is entirely novel for all members of the class. During the Low Lecture trial (see Guide #2 in this series) we gave students novel situations to explore as an attempt to induce semi-authentic mathematical activity. The students were required to “make up” new mathematical objects and explore their properties. In subsequent discussion, there were instances of abstraction, generalising, hypothesising, and justifying, amongst other habits.

We did not make any formal attempts to observe and document the quality of any habits, but we are convinced that an activity of this type, if accepted as a valuable part of an undergraduate course, would be a rich source of observations.

A third technique is to induce the habit within a topic and at a level that is part of the pre-requisite knowledge for the present course. While this is a seemingly attractive option, it has two drawbacks. First, it does not completely solve the problem as more advanced students can still bring their advanced knowledge to bear. Second, most mathematical habits are activated when the person is at the edge of their mathematical knowledge, thus any habits exhibited will be simplified shadows of what might be possible.

Observing mathematical habits

In this section, we briefly describe our attempts to observe two particular mathematical habits, and give our recommendations for others to either build on our attempts or avoid the pitfalls we met.

Persisting

The first task was to define persistence in a way that was observable. Some literature uses persistence with a negative connotation and perseverance for the positive attribute. We wish to encompass both, with a focus on behaviour with respect to a mathematical task. The diagram below was developed after much discussion:

Our attempt to capture the level and nature of

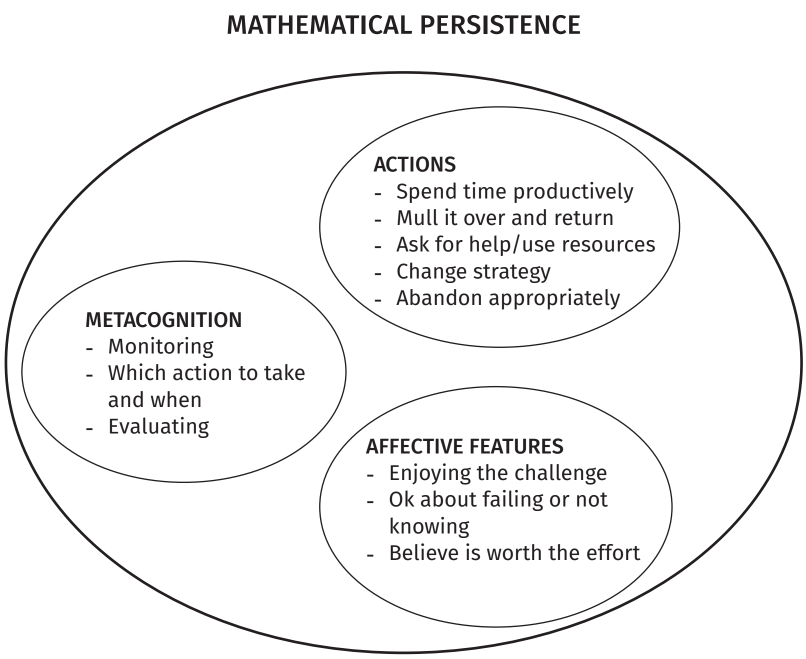


Figure 1: A Model of the components of Mathematical Persistence

persistence in a 100-level undergraduate class took the form of a questionnaire that was attached to an assignment (see Appendix 1). In our trialling we identified at least three problems with this instrument.

First, and most critically, we found that both our discussions, and students' responses to the questionnaire, conflated persistence with problem-solving. The two are difficult to separate because problem-solving is the context for persistence.

Second, we found that student responses to different questions were sometimes contradictory, and were always very context dependent. When responding, the student seemed to focus on one aspect of the assignment; hence, with a different question, a different part of the assignment was in focus and a different response was given.

Finally, persistence does not mean continuous work on a problem. Thus, the time scale can be very different for different instances of persistence. We did not manage to capture this aspect of the habit.

Mathematical Foresight

Mathematical Foresight was a habit identified by Wes Maciejewski in 2015. With Wes, we explored this idea in interviews with mathematicians and arrived at a description and a model (Maciejewski & Barton, 2016).

We then attempted to construct Foresight Eliciting Tasks—mathematical tasks that would induce foresight in students. We trialled three tasks: the first asked students to construct a function with certain characteristics; the second asked students to describe the likely characteristics of a mathematical model of algal blooms on beaches in summer; and the third asked students to suggest probabilities of winning a particular game.

Students' responses to these tasks were examined for the following factors:

1. Response did not address foresight
2. Listed factors relevant to the resolution
3. Indicated how these factors interact
4. Gave a mathematical relationship between the factors
5. Indicated consequences of the relationships
6. Stated limitations/strengths/generalisations of the chosen approach

Apart from factor 5 (which was not present in any response), there was a good spread of responses. That is, the class of thirty students exhibited all stages in their responses. The problem was that there was little consistency across the three different tasks, and we did not, overall, feel that we had captured the essence of mathematical foresight with this scheme.

We believe the difficulty lay in the task rather than the marking scheme. We realised that we needed a lot more experience at setting tasks.

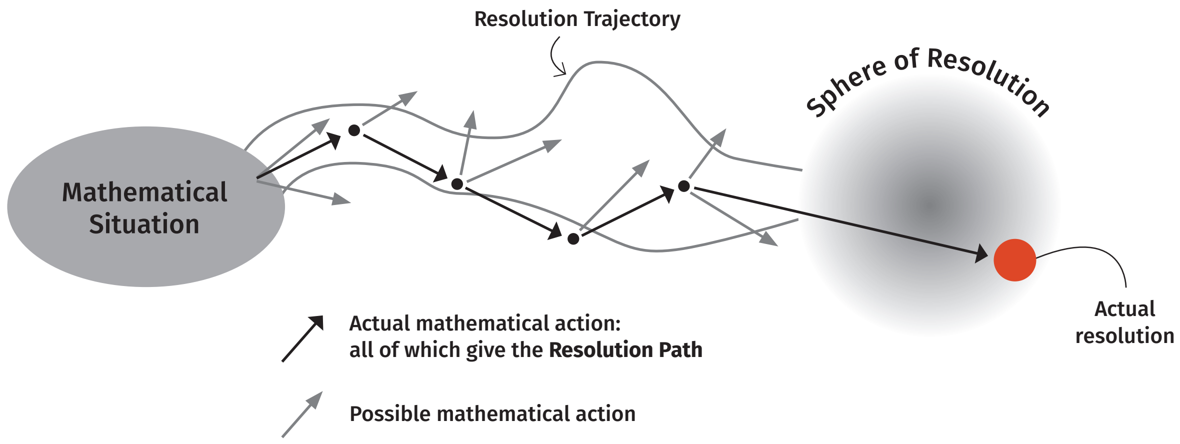


Figure 2: Mathematical Foresight: being able to describe in advance the general direction of the Resolution Trajectory and shape of the Sphere of Resolution.

Other habits

Within the LUMOS Project, an international group of lecturers began discussions on how to elicit hypothesising and generalising behaviours so that they may be observed. As with the foresight habit described above, the key issue is to construct tasks that will reliably produce the target behaviour in a form that can be observed. Furthermore, the observation needs to be robust enough to see the habit behaviour develop over a period of time. We did not achieve this aim, although we found considerable inspiration in the work of Al Cuoco www.edc.org/al-cuoco (and see Cuoco, Goldenburg, and Mark, 1996).

Both proving and mathematical modelling have extensive literatures that are easily found. Indeed, mathematical modelling has a whole community of mathematicians and mathematics educators, with its own conference cycle, the International Community of Teachers of Mathematical Modelling and Applications (ICTMA) www.ictma15.edu.au. We did not make any significant progress beyond this literature.

Table 1: A Catalogue of Habits

Category	***Habits***	***Sub-habits***
Mathematical Meta-habits	Abstracting	Objectifying
	Generalising	
	Reasoning	Analysing and synthesising
	Modelling	
	Problem-solving	
Argumentation	Testing	Finding examples and counter-examples
	Justifying Convincing	
	Proving	
Mathematical Habits	Foresight	
	Identifying …	… key elements … structure … variance and invariance … assumptions … limitations and boundary conditions
	Conjecturing Hypothesising	
	Symbolising	
	Defining	Refining definitions Exploring consequences
	Changing representation	
Mathematical Working Habits	Finding a suitable problem	
	Playing Exploring Experimenting	
	Persisting Perseverance “Sustained niggling”	… and knowing when to stop.
	Confidence …	… not to be intimidated … to make mistakes
	Working with others	

Notes

1. Mathematical communication is taken as a separate item and is not listed in this catalogue.
2. Some terms used here have different meanings in some literature. For example, argumentation is sometimes regarded as an early form of proving, whereas we use the term as a general category.
3. We regard this list as incomplete, and not necessarily properly structured. We welcome comments and additions.

Appendix 1

Persisting Questionnaire

Please think of one task/question/problem in this assignment that you found particularly difficult. (If none of them were difficult please recall another task instead.)

1. Please state the task/question/problem you are thinking of:

2. When you first saw this task/question/problem, did you know how to complete it?

   ☐ No ☐ Yes

3. Did you think you would be able to complete it **eventually**?

   ☐ No ☐ Unsure ☐ Yes

4. Did you seek help/further knowledge in trying to complete it? (Select all that apply)

   ☐ No ☐ Yes: ☐ Lecture notes ☐ Text book ☐ Library resources ☐ Internet
   ☐ Classmate ☐ Another student ☐ Tutor ☐ Lecturer

   Other (please specify):

5. Did you use a variety of problem solving strategies, such as: examining the question carefully; identifying assumptions about the task; linking it to something you knew; modifying a similar problem or task; breaking it down to parts or smaller problems; using an analogy; identifying tools and techniques needed to solve it; trying out a tool or technique you had learned; working backwards;
   Other (please specify):

   ☐ None ☐ One of these ☐ Some of these ☐ Many of these

6. In addition to the above, did you: (Select all that apply)

   ☐ Seek out resources to help ☐ Start again and try a different approach
   ☐ Take a break and come back to it again later ☐ Invest a lot of time on the problem

7. Please explain your answers to question 6. How did you know that it was a good idea to do this?

8. How would you rate your level of persistence on this task?

   ☐ Low (For example, you gave up immediately) ☐ Low to mid
   ☐ Mid (For example, you made a number of attempts) ☐ Mid to high
   ☐ High (For example, you kept trying different strategies until you had completed it)

9. Can you explain why? – why did you give up / why did you keep persisting?

10. How did it feel when you solved the problem / or gave up?

11. How would you rate your **usual** level of persistence on tasks given in this course?

☐ Low ☐ Low to mid ☐ Mid ☐ Mid to high ☐ High

12. If your answer to Q11 was different to Q8 can you explain why?

13. How would you rate your usual level of persistence on tasks given in **other** courses?

☐ Low ☐ Low to mid ☐ Mid ☐ Mid to high ☐ High

14. If you answer to Q13 was different to Q11 can you explain why?

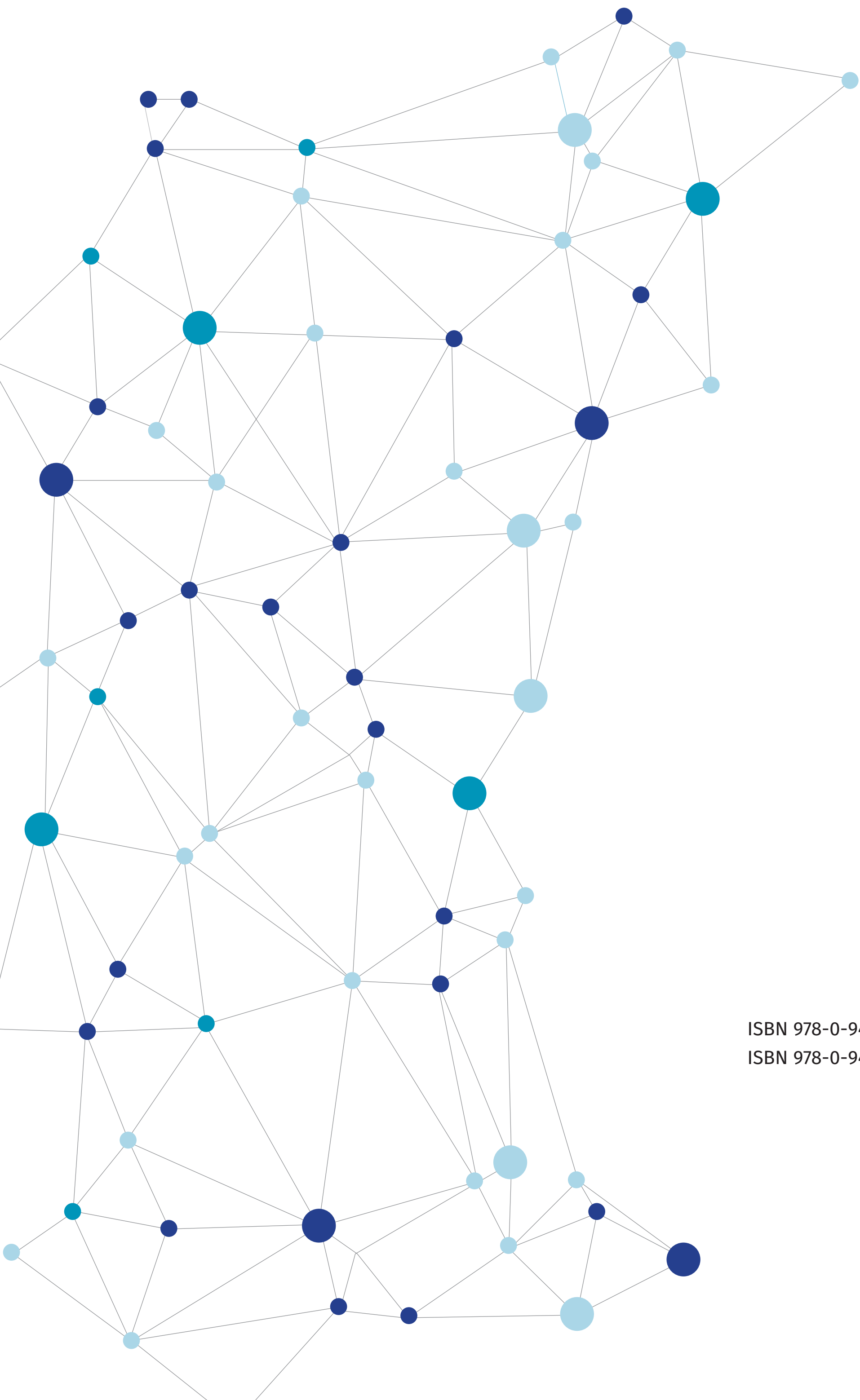
References

Cuoco, A., Goldenberg, P., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *Journal of Mathematical Behavior*, 15, 375–402

Maciejewski, W. & Barton, B. (2016). A Framework for Students' Mathematical Foresight. Presentation at ICME 13, Hamburg.

Polya, G. (1962). Mathematical discovery: On understanding, learning, and teaching problem solving. New York, NY: Wiley.

Watson, A. & Barton, B. (2011). Teaching mathematics as the contextual application of mathematical modes of enquiry. In K. Ruthven & T. Rowlands (Eds.), *Mathematical Knowledge in Teaching*, pp65-82, Cambridge University Press.



ISBN 978-0-947516-59-8 (online)
ISBN 978-0-947516-60-4 (print)